Pigeonhole Principle

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If n pigeons are assigned to m pigeonhales and $m < n$, then there is atleast one pigeonhole that contain two or more pigeons.

<u>Pf</u>:→ Let us label the n pigeons with the numbers 1 to n and n pigeonhales with the numbers from 1 to m.
Now starting with pigeon 1 and pigeonhale 1, assign each pigeon in order to the pigeonhole with the same number.

pigenholes O O O - - - - - O
1 2 3 m

pigeons 1 2 3 . . - - m $\underbrace{- - - - - - - - n}_{n-m}$

So we can assign as many pigeons as possible to distinct pigeonhales, but we know that pigeonhales are less than pigeons i.e $m < n$.
Thus there remain $n-m$ pigeons that have yet been assigned to pigeonhole. Hence, there is atleast one pigeonhole that will be assigned a second pigeon.

Q:→ Show that atleast two people must have their birthday in the same month if 13 people are assembled in a room.

sol:→ We assigned each person the month of the year on which he/she was born. Since there are 12 months in a year. By Pigeonhole principle, there are two people with birthday in the same month.

As No. of pigeons (people) $n = 13$

No. of pigeonholes (months) $m = 12$

$$n > m$$

Q:→ Show that if seven numbers from 1 to 12 are chosen, then two of them will add upto 13.

sol:→ Make sets, each containing two numbers from 1 to 12 whose sum is 13

I) 1, 12 II) 2, 11 III) 3, 10 IV) 4, 9 V) 5, 8 VI) 6, 7

Seven number chosen must belong to one of these sets. As there are only six sets, two of the chosen numbers belong to the same whose sum is 13.

Q:→ Show that if any 8 positive integers are chosen, two of them will have same remainder when divided by 7.

sol:→ Let 'a' be any number which is divided by 7 then

$$a = 7q + r \; ; \quad r = 0, 1, 2, 3, 4, 5, 6$$

Acc. to the Pigeonhole principle

where no. of positive integers (pigeons) $n = 8$

no. of remainder (pigeonhole) $m = 7$

$$m < n$$

Two numbers have same remainder.

<u>Extended Pigeonhole Principle</u> :→

If n pigeons are assigned to m pigeonholes where $n >> m$ (n is sufficiently large than m), then one of the pigeonhole must contain atleast $\left[\frac{n-1}{m}\right] + 1$ pigeons.

Pf:→ We prove this principle by contradiction.

Assume that each pigeonhole contain less than or equal to $\left\lfloor \frac{n-1}{m} \right\rfloor$ pigeons.

There are total m pigeonholes

$\therefore$ No. of pigeons in these m pigeonholes are

$$\leq m \left\lfloor \frac{n-1}{m} \right\rfloor$$

$$\leq m \left(\frac{n-1}{m} \right) \qquad [x] \leq x$$

$$= n-1 \text{ pigeons} \qquad (\rightarrow\leftarrow)$$

But there are total n pigeons

Hence, for given m pigeonholes, one of the pigeonhole must contain atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

Q:→ Show that if 9 colours are used to paint 1000 houses atleast 112 houses will be of the same colour.

sol:→ Consider houses as pigeons and colours as pigeonholes

No. of houses $(n) = 1000$

No. of colours $(m) = 9$

$$n >> m$$

Acc to Extended Pigeonhole Principle

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{1000-1}{9} \right\rfloor + 1 = \left\lfloor \frac{999}{9} \right\rfloor + 1 = 111 + 1 = 112$$

$\therefore$ There are 112 houses of same colour.

Q:→ How many people among 200000 are born at the same time (hour, minute, second). Use pigeonhole principle to find it.

sol:→ Total no. of people = 200000

Total no. of seconds in a day = $24 \times 60 \times 60$

= 86,400 seconds

Here, we're to assign time as pigeonhole.

No. of pigeons (n) = 200000

No. of pigeonhole (m) = 86,400

Acc. to Pigeonhole Principle.

Min no. of persons having same birthday time

$$= \left[\frac{n-1}{m}\right] + 1$$

$$= \left[\frac{200000-1}{86400}\right] + 1$$

$$= 2+1$$

$$= 3$$

Q:→ Each student of a class of 27 students go swimming on some of the days from Monday to Fridays in a certain week. If each student goes atleast twice then show that there must be atleast two students who go swimming on exactly the same days.

sol:→ No. of days for swimming = 5

The set S = {Mon, Tue, Wed, Fri} at 5 days has

${}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 26$ subsets each containing 2 or more days

If we treat 27 students as pigeons and these 26 students as pigeonholes then by pigeonhole principle, atleast two pigeons i.e students must go for swimming on the same days.